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Journal of Sound and Vibration 322 (2009) 524–531
JOURNAL OF SOUND AND VIBRATION
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# Energy transfer in weakly coupled nonlinear oscillator chains: Transition from a wandering breather to nonlinear self-trapping

Yuriy A. Kosevich*, Leonid I. Manevitch, Alexander V. Savin

*Semenov Institute of Chemical Physics, Russian Academy of Sciences, ul. Kosygina 4, 119991 Moscow, Russia*

Received 15 April 2008; received in revised form 28 May 2008; accepted 2 June 2008
The peer review of this article was organised by the Guest Editor
Available online 7 July 2008

**Abstract**

We present analytical and numerical studies of phase-coherent dynamics of intrinsically localized excitations (breathers) in a system of two weakly coupled nonlinear oscillator chains. We show that there are two qualitatively different dynamical regimes of the coupled breathers, either immovable or slowly moving: the periodic wandering of the low-amplitude breather between the chains, and the one-chain-localization of the high-amplitude breather. These two modes of coupled breathers can be mapped exactly onto two solutions of a pendulum equation, detached by a separatrix mode. We also show that these two regimes of the coupled breathers are similar, and are described by a similar pair of equations, to the two regimes in the nonlinear tunneling dynamics of two weakly coupled Bose–Einstein condensates. On the basis of this analogy, we predict a new tunneling mode of two weakly coupled Bose–Einstein condensates in which their relative phase oscillates around $\pi/2$ modulo $\pi$.
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## 1. Introduction

Nonlinear excitations (solitons, kink-solitons, intrinsically localized modes and breathers) can be created most easily in low-dimensional (one-dimensional (1D) and quasi-1D) systems [1–9]. Recent experiments have demonstrated the existence of intrinsically localized modes and breathers in various systems such as coupled nonlinear optical waveguides [10], low-dimensional crystals [11], antiferromagnetic materials [12], Josephson junction arrays [13,14], photonic structures and micromechanical systems [15], $\alpha$-helices [16] and proteins [17], and $\alpha$-uranium [18]. Slowly-moving breathers and supersonic kink-solitons were also described in 1D nonlinear chains [6,7,19–21], DNA macromolecules [22] and quasi-1D polymer crystals [23].

One-dimensional arrays of magnetic or optical microtraps for Bose–Einstein condensates (BECs) of ultracold quantum gases with tunneling coupling provide a new field for the studies of coherent nonlinear dynamics in low-dimensional systems [24,25]. In the mean-field theory, the tunneling coupling between two BECs is similar to the linear coupling between two optical waveguides [26] or two oscillator chains (nonlinear

*Corresponding author. Tel.: +7 495 939 7235; fax: +7 495 137 8284.
*E-mail addresses:* yukosevich@yahoo.com (Yu.A. Kosevich), lmanev@center.chph.ras.ru (L.I. Manevitch), asavin@center.chph.ras.ru (A.V. Savin).

0022-460X/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2008.06.005

phononic waveguides). In this paper, we show that *phase-coherent* dynamics of macroscopic ensembles of classical particles in two weakly linked nonlinear oscillator chains has a profound analogy, and is described by pair of equations, similar in every respect to the tunneling quantum dynamics of two weakly linked interacting (nonideal) BECs in a macroscopic double-well potential (single bosonic Josephson junction) [27]. The exchange of energy and excitations between the coupled classical oscillator chains takes on the role which the exchange of atoms via quantum tunneling plays in the case of coupled BECs. Therefore such phase-coherent energy and excitation exchange can be considered as a classical counterpart of macroscopic tunneling quantum dynamics.

We show that there are two qualitatively different dynamical regimes of the coupled breathers, the oscillatory exchange of the low-amplitude breather between the chains (*wandering breather*), and one-chain-localization (nonlinear self-trapping) of the high-amplitude breather. These two regimes, which are detached by a separatrix mode with zero rate of energy and excitation exchange, are analogous to the two regimes in nonlinear dynamics of macroscopic condensates in a single bosonic Josephson junction [27]. Essentially the phase-coherent dynamics of the coupled classical breathers is described by a pair of equations, which coincides with the pair of coupled mean-field equations describing coherent atomic tunneling in a single bosonic tunnel junction [28,29]. The predicted evolution of the relative phase of the two weakly coupled coherent breathers in both regimes is also analogous to the evolution of relative quantum mechanical phase between two weakly coupled macroscopic condensates, which was directly measured in a single bosonic Josephson junction by means of interference [27]. All these results bring to light a striking similarity, both in display and evolution equations, between the classical phase-coherent excitation exchange and macroscopic tunneling quantum dynamics which can motivate new predictions and experiments in both fields. For instance, we predict a new tunneling regime of two coupled BECs in which their relative phase oscillates around $\pi/2$ modulo $\pi$, which can be observed by means of interference. This regime is different from the regime of Josephson oscillations, realized in experiments [27], when the relative phase of two weakly coupled BECs oscillates (or fluctuates [30]) around zero (modulo $2\pi$). The obtained here dispersion and evolution equations and the form of the coupled phase-coherent nonlinear excitations can be applied both to wandering macroscopic Bose–Einstein condensate, slowly moving along two weakly linked bosonic waveguides, and to wandering macroscopic (weakly localized) breather, slowly moving along two weakly linked macromolecules, $\alpha$-helices or DNA.

## 2. Model and analytical predictions

We consider two linearly coupled nonlinear oscillator chains (with unit lattice period and unit mass). Each chain we model with the Fermi–Pasta–Ulam ($\beta$-FPU) Hamiltonian, which is one of the most simple and universal models of nonlinear lattices and which can be applied to a diverse range of physical problems [31]:

$$H = \sum_n \left[ \sum_{i=1}^{2} \left[ \frac{1}{2} p_n^{(i)2} + \frac{1}{2} k^{(i)} (u_{n+1}^{(i)} - u_n^{(i)})^2 + \frac{1}{4} \beta (u_{n+1}^{(i)} - u_n^{(i)})^4 \right] + \frac{1}{2} C (u_n^{(1)} - u_n^{(2)})^2 \right], \tag{1}$$

where $u_n^{(i)}$ is displacement of the $n$th particle from its equilibrium position in the $i$th chain, $p_n^{(i)} = \dot{u}_n^{(i)}$ is particle momentum, $k^{(i)}$, $\beta$ and $C$ are, respectively, intra-chain linear, nonlinear and inter-chain linear force constants. We assume that the coupling is weak, $C \ll 1$, and therefore do not include the nonlinear inter-chain interaction. The $\beta$-FPU Hamiltonian (1) describes, e.g., purely transverse particle motion [6]. The torsion dynamics of DNA double helix can also be approximated by two weakly coupled $\beta$-FPU chains [22].

We are interested in high-frequency and therefore short-wavelength dynamics of the coupled chains, when the displacements of the nearest-neighbor particles are mainly antiphase. For this case we introduce continuous envelope-functions for the particle displacements in the chains, $f_n^{(i)} = u_n^{(i)}(-1)^n$ and $f_n^{(i)} \equiv f(x)_i$. These envelope-functions $f(x)_i$ are supposed to be slowly varying on the interatomic scale in both chains, $\partial f_i/\partial x \ll 1$, which allows us to write corresponding partial differential equations for these functions; see, e.g., Refs. [2,6,20,32]. Then from Eq. (1) we get the following equations for $f(x)^{(i)}$, $i = 1, 2$:

$$\ddot{f}^{(i)} + \omega_{mi}^2 f^{(i)} + \frac{\partial^2 f^{(i)}}{\partial x^2} + 16\beta f^{(i)3} - C f^{(3-i)} = 0, \tag{2}$$

where $\omega_{mi} = \sqrt{4k^{(i)} + C}$ is characteristic frequency above the maximal phonon frequency in the $i$th isolated chain.

In order to deal with the amplitude and phase of the coupled nonlinear excitations, it is useful to introduce complex wave fields $\Psi(x,t)_i$ for each chain, cf. Ref. [32]:

$$f(x,t)^{(i)} = \tfrac{1}{2}[\Psi(x,t)_i + \Psi(x,t)_i^*]. \tag{3}$$

Then from Eqs. (2) and (3) we get the following coupled equations for $\Psi(x,t)_i$, $i = 1,2$:

$$\frac{1}{2}\left(\frac{\partial^2\Psi_i}{\partial t^2} + \frac{\partial^2\Psi_i}{\partial x^2} + \omega_{mi}^2\Psi_i\right) + 6\beta|\Psi_i|^2\Psi_i = \frac{C}{2}\Psi_{3-i} \tag{4}$$

and complex-conjugated equations for $\Psi_i^*$. To describe a slowly moving breather, wandering between two weakly coupled nonlinear chains with positive (repulsive) anharmonic force constant $\beta$, we assume the following form for the complex fields $\Psi_1$ and $\Psi_2$:

$$\Psi_1 = \Psi_{\max}\frac{\exp[\mathrm{i}(kx-\omega t)]}{\cosh[\lambda_1(x-Vt)]}\cos\Theta\exp\left(-\frac{\mathrm{i}}{2}\Delta\right), \tag{5}$$

$$\Psi_2 = \Psi_{\max}\frac{\exp[\mathrm{i}(kx-\omega t)]}{\cosh[\lambda_2(x-Vt)]}\sin\Theta\exp\left(\frac{\mathrm{i}}{2}\Delta\right), \tag{6}$$

where $\omega$, $V \ll 1$ and $k \ll 1$ are, respectively, frequency, slow velocity and small wavenumber related with the moving breather, $\lambda_i$ describe inverse localization lengths; $\Delta = \Delta(t - kx/\omega)$ stands for the relative phase of the coupled chains, while the parameter $\Theta = \Theta(t - kx/\omega)$ describes the "relative excitation" (excitation imbalance) of the two chains $z = (n_1 - n_2)/(n_1 + n_2) = \cos 2\Theta$, where $n_i = |\Psi_i|^2$ is local density of excitations in the $i$th chain, cf. Refs. [28,29].

Using Eqs. (4), (5) and (6), after some algebra we obtain dispersion equations for the introduced parameters,

$$\omega^2 = \frac{1}{2}(\omega_{m1}^2 + \omega_{m2}^2) + 3\beta\Psi_{\max}^2 - k^2 - C\frac{\cos\Delta}{\sin(2\Theta)},$$

$$\lambda_1^2 = 6\beta\Psi_{\max}^2\cos^2\Theta, \lambda_2^2 = 6\beta\Psi_{\max}^2\sin^2\Theta, V = \frac{\partial\omega}{\partial k} \tag{7}$$

and evolution equations for $\Theta$ and $\Delta$:

$$\dot{\Theta} = \frac{C}{2\omega}\sin\Delta, \tag{8}$$

$$\dot{\Delta} = \frac{1}{2\omega}(\omega_{m1}^2 - \omega_{m2}^2) + \frac{3\beta\Psi_{\max}^2}{\omega}\cos(2\Theta) + \frac{C}{\omega}\cos\Delta\cot(2\Theta). \tag{9}$$

In the derivation of Eqs. (8) and (9), it was assumed explicitly that the ratio $\cosh[\lambda_1(x-Vt)]/\cosh[\lambda_2(x-Vt)]$ is equal to one. The latter is valid for small-amplitude breathers with long localization lengths, $\lambda_{1,2} \ll 1$. In this case the above assumption, which is exact for the central region of the breathers, $x - Vt \approx 0$, will be (approximately) valid for a large number of particles, which form weakly localized (macroscopic) wandering breather in weakly coupled nonlinear chains. Eqs. (8) and (9) can be written in an equivalent form for the relative excitation $z$ and relative phase $\Delta$, when $z = \cos 2\Theta$ and $\sqrt{1-z^2} = \sin 2\Theta$:

$$\dot{z} = -\frac{C}{\omega}\sqrt{1-z^2}\sin\Delta, \tag{10}$$

$$\dot{\Delta} = \frac{1}{2\omega}(\omega_{m1}^2 - \omega_{m2}^2) + \frac{3\beta\Psi_{\max}^2}{\omega}z + \frac{C}{\omega}\frac{z}{\sqrt{1-z^2}}\cos\Delta. \tag{11}$$

Here the variables $z$ and $\Delta$ are canonically conjugate, $\dot{z} = -\partial H_{\mathrm{eff}}/\partial\Delta$, $\dot{\Delta} = \partial H_{\mathrm{eff}}/\partial z$, with the effective Hamiltonian $H_{\mathrm{eff}} = 3\beta\Psi_{\max}^2/2\omega z^2 - C/\omega\sqrt{1-z^2}\cos\Delta + z/2\omega(\omega_{m1}^2 - \omega_{m2}^2)$.

The very same Eqs. (10) and (11) for $z$ and $\Delta$, which are equivalent to Eqs. (8) and (9) for $\Theta$ and $\Delta$, were derived in Refs. [28,29] in the mean-field theory of quantum coherent atomic tunneling and coherent oscillations between two weakly coupled BECs, which were later used in the analysis of the experimental realization of a single bosonic Josephson junction [27]. In our case, generic evolution Eqs. (10) and (11) describe the exchange of lattice excitations between the (nonidentical in general) chains rather than the exchange of atoms via quantum tunneling.

It is noteworthy that equations, similar to Eqs. (8) and (9), describe the dynamics of two weakly coupled nonlinear oscillators. Therefore wandering breathers can be considered as weakly coupled phase-coherent nonlinear *macroscopic oscillators*.

For two *identical* chains, $\omega_{m1} = \omega_{m2} \equiv \omega_m$, the ansatz

$$\cos \Delta = A(t)/\sin(2\Theta), \tag{12}$$

where $A = 0$ for $\sin(2\Theta) = 0$, gives us from Eqs. (8) and (9) that

$$\cos \Delta = -\frac{3\beta\Psi_{\max}^2}{2C}\sin(2\Theta) = -\frac{3\beta\Psi_{\max}^2}{2C}\sqrt{1-z^2}. \tag{13}$$

This solution, being exact solution of Eqs. (8) and (9), (10) and (11), conserves the effective Hamiltonian: $H_{\text{eff}} = 3\beta\Psi_{\max}^2/2\omega$. The important feature of this solution is that the relative phase $\Delta$ is self-locked to the value $\pi/2$ modulo $\pi$ by the total excitation imbalance $z = \pm 1$ of the two coupled chains. The phase portrait of Eq. (13) on the $z-\Delta$ plane is given by $(\kappa \cos \Delta)^2 + z^2 = 1$, where $\kappa = 2C/3\beta\Psi_{\max}^2$, see Fig. 1.

Finally, for two identical weakly coupled chains with $k = 1$ and $\omega_m = 2$ we get from Eqs. (8), (9) and (13) the nonlinear physical-pendulum equation for $\Xi = 4\Theta$:

$$\ddot{\Xi} + \Omega_0^2 \sin \Xi = 0, \tag{14}$$

where $\Omega_0 = 3\beta\Psi_{\max}^2/4$. We solve this equation with the initial condition $\Theta(0) = 0$, which corresponds to zero complex field $\Psi_2$ in the second chain at $t = 0$ (or $z(0) = 1$), and which is realized in our numerical simulations. Therefore we assume that $\Xi(0) = 0$, $\Delta(0) = \pi/2$ and $\dot{\Xi}(0) = C$. The general solution of Eq. (14) is well known and can be written in terms of elliptic functions, with the elliptic modulus $\kappa = 2C/3\beta\Psi_{\max}^2$. The case of $\kappa = 1$ corresponds to the separatrix between the two qualitatively different dynamical regimes of the pendulum equation (14).

For $\kappa \gg 1$ or $\beta\Psi_{\max}^2 \ll \frac{2}{3}C$, $\Theta$ linearly grows with the "running" time $\tilde{t} \equiv t - (k/\omega)x$, while $\Delta \approx \Delta(0) = \pi/2$:

$$\Theta \approx (C - \frac{\Omega_0^2}{C})\frac{\tilde{t}}{4}, \quad \Delta \approx \frac{\pi}{2} + \frac{3\beta\Psi_{\max}^2}{2C}\sin\left[\left(C - \frac{\Omega_0^2}{C}\right)\frac{\tilde{t}}{2}\right]. \tag{15}$$

In this regime $\Theta$ spans the full range from 0 to $2\pi$, which corresponds to the *complete* excitation exchange between the nonlinear chains and therefore to the breather, wandering between the two chains. The rate of the

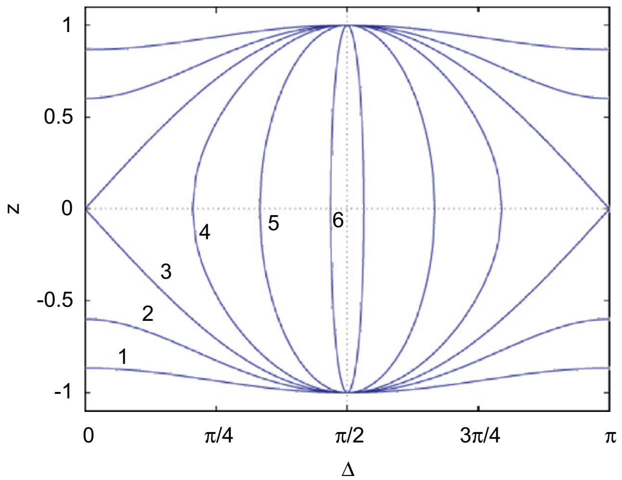


Fig. 1. Phase portrait (relative excitation $z$ versus relative phase $\Delta$) of a breather in two coupled nonlinear chains, which describes the solution $(\kappa \cos \Delta)^2 + z^2 = 1$, $\kappa = 2C/3\beta\Psi_{\max}^2$, given by Eq. (13). Lines 1, 2, 3, 4, 5, and 6 correspond to $k = 0.5, 0.8, 1, 1.25, 2$ and 10. Lines 1 and 2 describe the self-trapped mode, lines 4–6 describe the wandering breather, line 3 describes the separatrix.

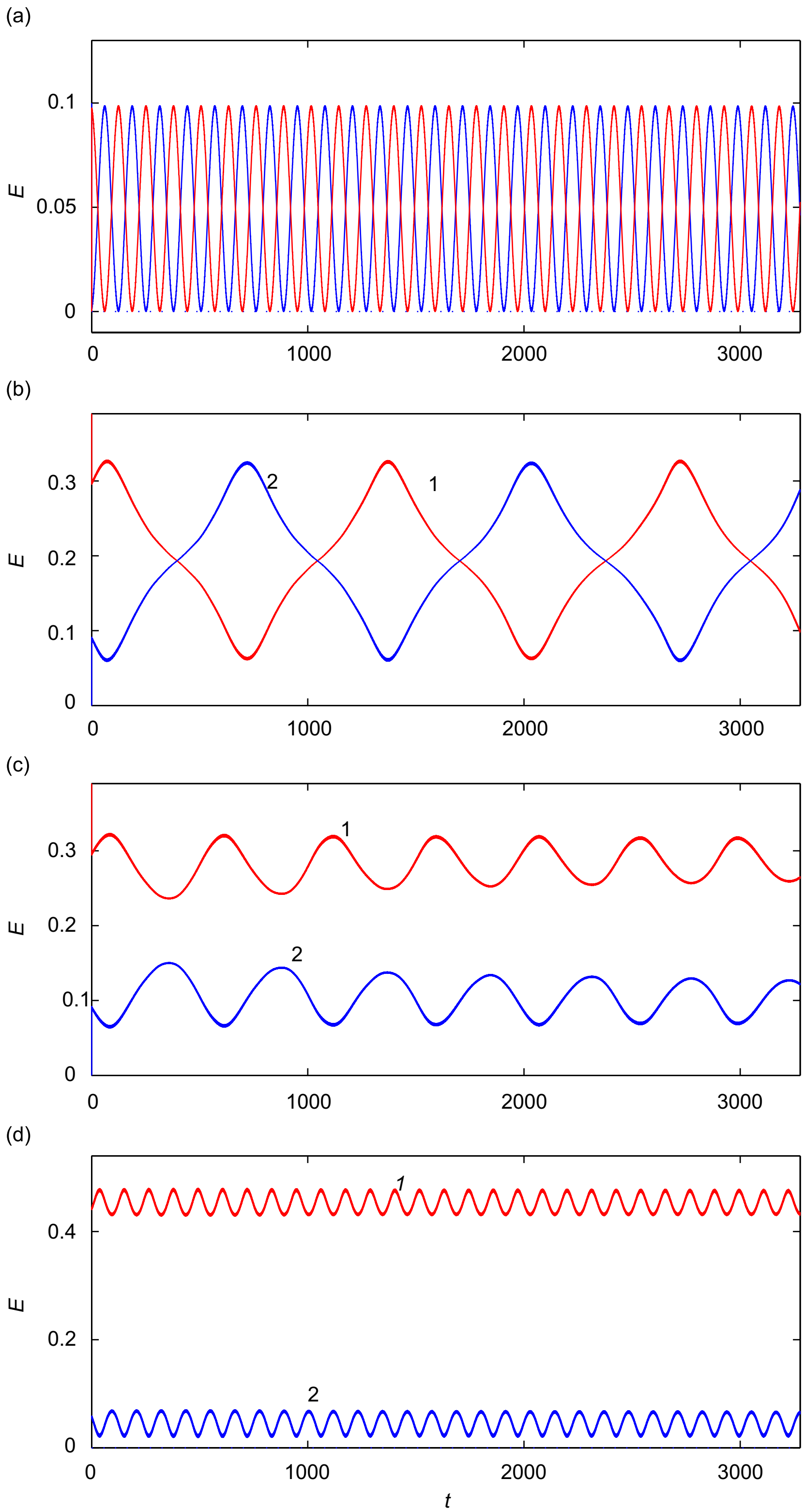


Fig. 2. Energy of immovable breather, with $V=0$, in chains 1 and 2 versus time, obtained from numerical solution of Eq. (1) for two coupled $\beta$-FPU chains with the initial breather excitation in chain 1 (with immovable chain 2) with frequency $\omega = 2.03$ (a), $\omega = 2.098$ (b), $\omega = 2.0981$ (c) and $\omega = 2.10$ (d). The chains with $\beta = 1$ and $C = 0.1$, and absorbing edges were used in simulations.

excitation exchange in this mode, $(C - \Omega_0^2/C)/2$ for $\kappa \gg 1$, continuously decreases with the increase of the ratio $\beta\Psi_{\max}^2/C$ below the separatrix. The solution $\Theta \approx C\tilde{t}/4$ and $\Delta \approx \pi/2$ can be obtained directly from the linearized coupled Eq. (4).

A similar tunneling regime can be also realized for the BEC in a double-well potential when the condensate is initially loaded into one of the wells, $z(0) = \pm 1$, cf. Ref. [26]. In such a regime the relative phase of the coupled BECs oscillates around $\pi/2$ modulo $\pi$, see Eq. (15), which can be observed by means of interference. This tunneling regime is similar, but also different from, the regime of Josephson oscillations, already realized in experiments [27], in which the relative population of two BECs (which is an equivalent of the relative excitation in two coupled oscillator chains) is always less than one and the relative phase of the coupled BECs oscillates around zero (modulo $2\pi$). In the opposite limit $\kappa \ll 1$ or $\beta\Psi_{\max}^2 \gg 2C/3$, one has

$$\Theta \approx \frac{C}{3\beta\Psi_{\max}^2}\sin\left(\frac{3}{4}\beta\Psi_{\max}^2\tilde{t}\right), \quad \Delta \approx \frac{\pi}{2} + \frac{3}{4}\beta\Psi_{\max}^2\tilde{t}. \tag{16}$$

This dynamical regime corresponds to the asymmetric nonlinear mode (known, e.g., for two coupled nonlinear waveguides [33–35]), in which one system, chain 1 here, carries almost all vibrational energy, while the other is almost at rest, and the energy exchange between the chains is relatively small. This dynamical regime is similar to the macroscopic quantum self-trapping of the BEC in a single bosonic Josephson junction [27]. The separatrix $\kappa = 1$ is characterized by zero rate of energy exchange and corresponds to a stationary solution of Eqs. (8), (9) and (13) (for $\Theta(0) = 0$, $\Delta(0) = \pi/2$):

$$\Theta = \pi/4, \quad \Delta = \pi, \quad \dot{\Theta} = \dot{\Delta} = 0. \tag{17}$$

## 3. Numerical simulations and comparison with analytical predictions

In Fig. 2 we show the energy of a breather in two weakly coupled chains versus time, which was computed from the numerical solution of Eq. (1) for two weakly coupled $\beta$-FPU chains for the breather excitation in chain 1 with frequency $\omega$ below, (a), very close, (b) and (c), and beyond, (d), the separatrix. Two identical chains with $k = 1$, $\beta = 1$ and $C = 0.1$, and absorbing edges were used in the simulations. The latter is necessary to get rid of weak radiation, caused by the wandering breather (and which will stay in the system forever in the case of periodic boundary conditions). For the chain parameters used, $\kappa = 1$ corresponds to $\omega = 2.1$ and $\Psi_{\max}^{(s)} = 0.2582$. A drastic decrease of the energy exchange rate is seen very close to the separatrix, plots (b) and (c) in comparison with (a) and (d).

We also obtain similar numerical results for the slowly moving wandering breather, see Fig. 3. Slowly moving wandering BECs (in weakly interacting limit) can be realized in two coupled bosonic waveguides with initial total population imbalance, $z(0) = \pm 1$. It is worth mentioning that the form and frequency of a breather in an *isolated* chain can be obtained only in the self-trapping breather mode. Indeed, according to Eqs. (7), (13) and (16), in the limit $C \to 0$ one has $\Theta = 0$ in Eqs. (5) and (6) and the breather frequency is equal to

$$\omega = \omega_m + \frac{3}{2}\beta\Psi_{\max}^2 - \frac{k^2}{4}. \tag{18}$$

This expression for the breather frequency is fully consistent with the known expressions for a single stationary or slowly moving breather in the small-amplitude limit, see, e.g., Refs. [2,32]. To get this expression for $\omega$, one has explicitly taken into account in Eq. (5) the linear increase in time (winding up) of the relative phase $\Delta$ in the self-trapping mode, given by Eq. (16). Similar winding up of the relative phase of two weakly coupled macroscopic BECs in the nonlinear self-trapping mode has been recently directly measured in a single bosonic Josephson junction by means of interference [27]. This finding gives us an additional argument in favor of the similarity between macroscopic tunneling quantum dynamics and phase-coherent dynamics of weakly coupled macroscopic breathers.

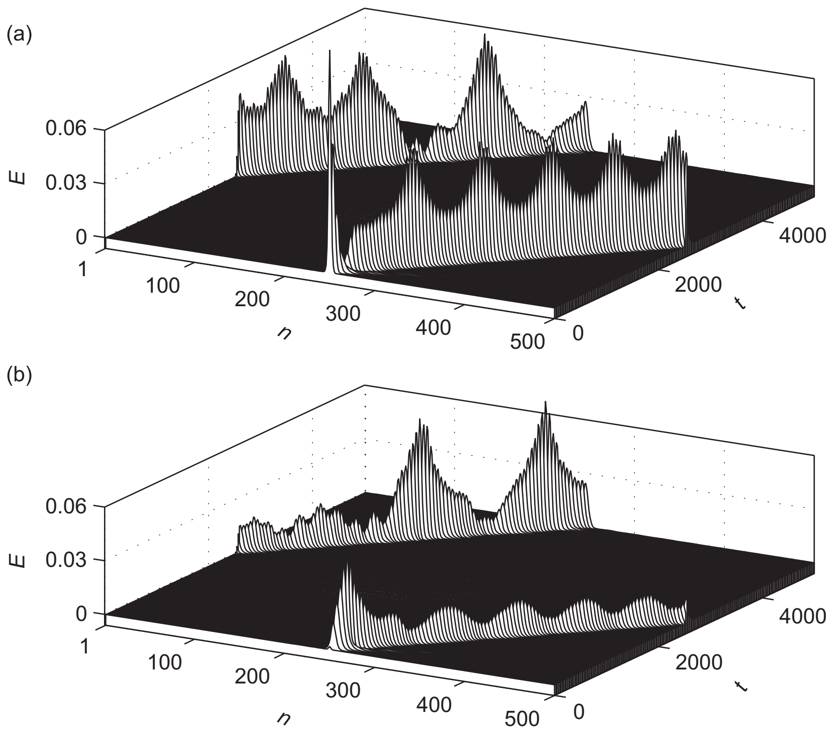


Fig. 3. Energy of slowly moving wandering breather close to the separatrix, versus time and site. The breather is initially excited in chain 1, (a) with immovable chain 2, (b) with velocity $V = 0.1$ and amplitude $\Psi_{\max} = 0.26$. The separatrix-like dynamics is well established for $t \geqslant 3000$ (cf. Fig. 2(b)).

## 4. Conclusions

In conclusion, we have found, both analytically and numerically, two qualitatively different regimes of energy exchange between phase-coherent breathers (macroscopic oscillators) in two weakly linked nonlinear oscillator chains. These regimes have a profound analogy, and are described by a similar pair of equations, to the anharmonic Josephson plasma oscillations and nonlinear self-trapping, recently observed in a single bosonic Josephson junction. The similarity between the classical phase-coherent excitation exchange and macroscopic tunneling quantum dynamics found here can encourage new experiments in both fields.

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